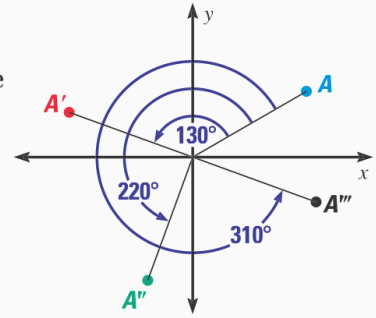


USE ROTATIONS

You can rotate a figure more than 360° . However, the effect is the same as rotating the figure by the angle minus 360° .

ROTATIONS ABOUT THE ORIGIN You can rotate a figure more than 180° . The diagram shows rotations of point A 130° , 220° , and 310° about the origin. A rotation of 360° returns a figure to its original coordinates.

There are coordinate rules that can be used to find the coordinates of a point after rotations of 90° , 180° , or 270° about the origin.



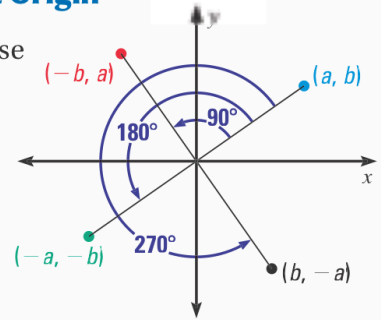
KEY CONCEPT

For Your Notebook

Coordinate Rules for Rotations about the Origin

When a point (a, b) is rotated counterclockwise about the origin, the following are true:

1. For a rotation of 90° , $(a, b) \rightarrow (-b, a)$.
2. For a rotation of 180° , $(a, b) \rightarrow (-a, -b)$.
3. For a rotation of 270° , $(a, b) \rightarrow (b, -a)$.



EXAMPLE 2 Rotate a figure using the coordinate rules

Graph quadrilateral $RSTU$ with vertices $R(3, 1)$, $S(5, 1)$, $T(5, -3)$, and $U(2, -1)$. Then rotate the quadrilateral 270° about the origin.

Solution

Graph $RSTU$. Use the coordinate rule for a 270° rotation to find the images of the vertices.

$$(a, b) \rightarrow (b, -a)$$

$$R(3, 1) \rightarrow R'(1, -3)$$

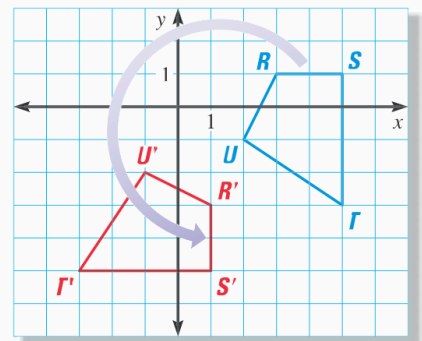
$$S(5, 1) \rightarrow S'(1, -5)$$

$$T(5, -3) \rightarrow T'(-3, -5)$$

$$U(2, -1) \rightarrow U'(-1, -2)$$

Graph the image $R'S'T'U'$.

 at classzone.com



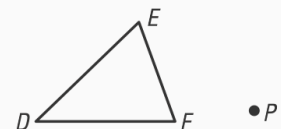
ANOTHER WAY

For an alternative method for solving the problem in Example 2, turn to page 606 for the **Problem Solving Workshop**.



GUIDED PRACTICE for Examples 1 and 2

1. Trace $\triangle DEF$ and P . Then draw a 50° rotation of $\triangle DEF$ about P . **See margin.**
2. Graph $\triangle JKL$ with vertices $J(3, 0)$, $K(4, 3)$, and $L(6, 0)$. Rotate the triangle 90° about the origin. **See margin.**



USING MATRICES You can find certain images of a polygon rotated about the origin using matrix multiplication. Write the rotation matrix to the left of the polygon matrix, then multiply.

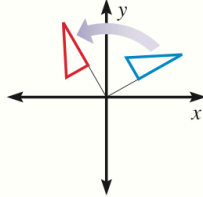
KEY CONCEPT

For Your Notebook

Rotation Matrices (Counterclockwise)

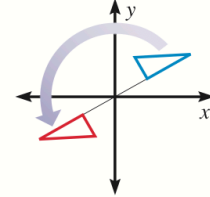
90° rotation

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



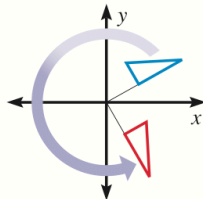
180° rotation

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



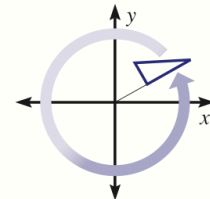
270° rotation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



360° rotation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



READ VOCABULARY

Notice that a 360° rotation returns the figure to its original position. Multiplying by the matrix that represents this rotation gives you the polygon matrix you started with, which is why it is also called the *identity matrix*.

EXAMPLE 3 Use matrices to rotate a figure

Trapezoid $EFGH$ has vertices $E(-3, 2)$, $F(-3, 4)$, $G(1, 4)$, and $H(2, 2)$. Find the image matrix for a 180° rotation of $EFGH$ about the origin. Graph $EFGH$ and its image.

Solution

STEP 1 Write the polygon matrix:

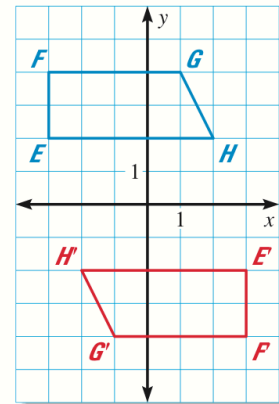
E	F	G	H
-3	-3	1	2
2	4	4	2

STEP 2 Multiply by the matrix for a 180° rotation.

E	F	G	H	$=$	E'	F'	G'	H'	
-1	0	-3	-3	1	2	3	3	-1	-2
0	-1	2	4	4	2	-2	-4	-4	-2

Rotation matrix	Polygon matrix	Image matrix
-----------------	-----------------------	---------------------

STEP 3 Graph the preimage $EFGH$. Graph the image $E'F'G'H'$.



AVOID ERRORS

Because matrix multiplication is not commutative, you should always write the rotation matrix first, then the polygon matrix.

GUIDED PRACTICE for Example 3

Use the quadrilateral $EFGH$ in Example 3. Find the image matrix after the rotation about the origin. Graph the image. **3–5. See margin.**

3. 90°

4. 270°

5. 360°

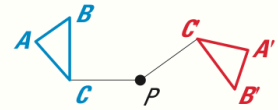
THEOREM

For Your Notebook

THEOREM 9.3 Rotation Theorem

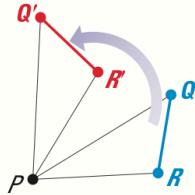
A rotation is an isometry.

Proof: Exs. 33–35, p. 604

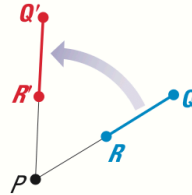


$$\triangle ABC \cong \triangle A'B'C'$$

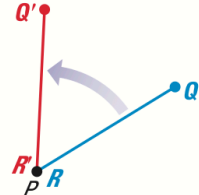
CASES OF THEOREM 9.3 To prove the Rotation Theorem, you need to show that a rotation preserves the length of a segment. Consider a segment \overline{QR} rotated about point P to produce $\overline{Q'R'}$. There are three cases to prove:



Case 1 R , Q , and P are noncollinear.



Case 2 R , Q , and P are collinear.



Case 3 P and R are the same point.

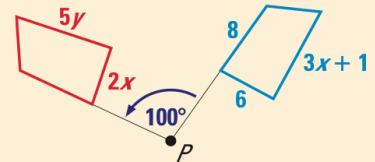


EXAMPLE 4

Standardized Test Practice

The quadrilateral is rotated about P .
What is the value of y ?

- (A) $\frac{8}{5}$ (B) 2
(C) 3 (D) 10



Solution

By Theorem 9.3, the rotation is an isometry, so corresponding side lengths are equal. Then $2x = 6$, so $x = 3$. Now set up an equation to solve for y .

$$5y = 3x + 1 \quad \text{Corresponding lengths in an isometry are equal.}$$

$$5y = 3(3) + 1 \quad \text{Substitute 3 for } x.$$

$$y = 2 \quad \text{Solve for } y.$$

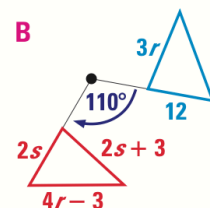
► The correct answer is B. (A) (B) (C) (D)



GUIDED PRACTICE for Example 4

6. Find the value of r in the rotation of the triangle. **B**

- (A) 3 (B) 5
(C) 6 (D) 15



9.5 Apply Compositions of Transformations



Before _____

Now _____

Why? _____ so you can describe the transformations that represent a ROWING

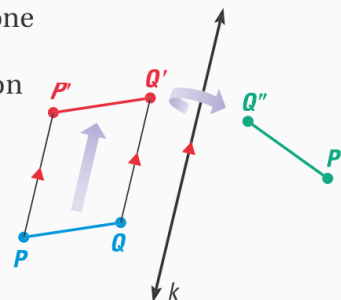
Key Vocabulary

- glide reflection
- composition of transformations

A translation followed by a reflection can be performed one after the other to produce a *glide reflection*. A translation can be called a glide. A **glide reflection** is a transformation in which every point P is mapped to a point P'' by the following steps.

STEP 1 First, a translation maps P to P' .

STEP 2 Then, a reflection in a line k parallel to the direction of the translation maps P' to P'' .



EXAMPLE 1 Find the image of a glide reflection

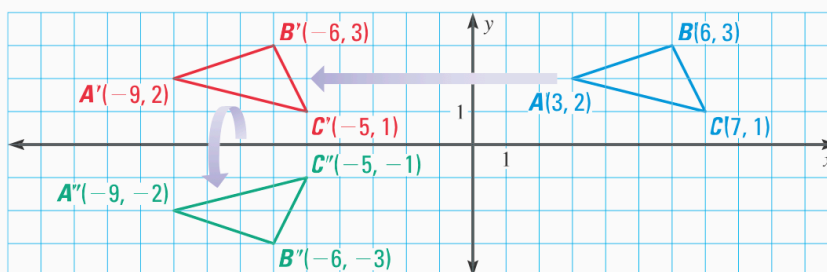
The vertices of $\triangle ABC$ are $A(3, 2)$, $B(6, 3)$, and $C(7, 1)$. Find the image of $\triangle ABC$ after the glide reflection.

Translation: $(x, y) \rightarrow (x - 12, y)$

Reflection: in the x -axis

Solution

Begin by graphing $\triangle ABC$. Then graph $\triangle A'B'C'$ after a translation 12 units left. Finally, graph $\triangle A''B''C''$ after a reflection in the x -axis.



AVOID ERRORS

The line of reflection must be parallel to the direction of the translation to be a glide reflection.

GUIDED PRACTICE for Example 1

- Suppose $\triangle ABC$ in Example 1 is translated 4 units down, then reflected in the y -axis. What are the coordinates of the vertices of the image?
 $A(-3, -2)$, $B(-6, -1)$, $C(-7, -3)$
- In Example 1, describe a glide reflection from $\triangle A''B''C''$ to $\triangle ABC$.
 $(x, y) \rightarrow (x + 12, y)$ followed by a reflection in y .

COMPOSITIONS When two or more transformations are combined to form a single transformation, the result is a **composition of transformations**. A glide reflection is an example of a composition of transformations.

In this lesson, a composition of transformations uses isometries, so the final image is congruent to the preimage. This suggests the Composition Theorem.

THEOREM

For Your Notebook

THEOREM 9.4 Composition Theorem

The composition of two (or more) isometries is an isometry.

Proof: Exs. 35–36, p. 614

EXAMPLE 2 Find the image of a composition

The endpoints of \overline{RS} are $R(1, -3)$ and $S(2, -6)$. Graph the image of \overline{RS} after the composition.

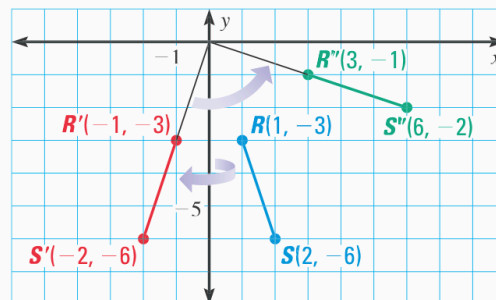
Reflection: in the y -axis
Rotation: 90° about the origin

Solution

STEP 1 Graph \overline{RS} .

STEP 2 Reflect \overline{RS} in the y -axis. $\overline{R'S'}$ has endpoints $R'(-1, -3)$ and $S'(-2, -6)$.

STEP 3 Rotate $\overline{R'S'}$ 90° about the origin. $\overline{R''S''}$ has endpoints $R''(3, -1)$ and $S''(6, -2)$.



AVOID ERRORS

Unless you are told otherwise, do the transformations in the order given.

TWO REFLECTIONS Compositions of two reflections result in either a translation or a rotation, as described in Theorems 9.5 and 9.6.

THEOREM

For Your Notebook

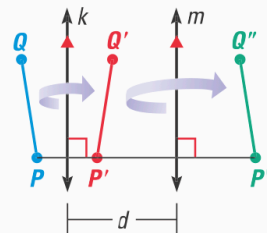
THEOREM 9.5 Reflections in Parallel Lines Theorem

If lines k and m are parallel, then a reflection in line k followed by a reflection in line m is the same as a translation.

If P'' is the image of P , then:

- $\overline{PP''}$ is perpendicular to k and m , and
- $PP'' = 2d$, where d is the distance between k and m .

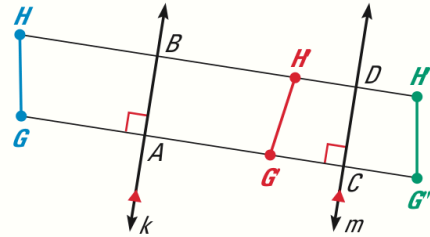
Proof: Ex. 37, p. 614



EXAMPLE 3 Use Theorem 9.5

In the diagram, a reflection in line k maps \overline{GH} to $\overline{G'H'}$. A reflection in line m maps $\overline{G'H'}$ to $\overline{G''H''}$. Also, $HB = 9$ and $DH'' = 4$.

- Name any segments congruent to each segment: \overline{HG} , \overline{HB} , and \overline{GA} .
- Does $AC = BD$? Explain.
- What is the length of $\overline{GG''}$?



Solution

- $\overline{HG} \cong \overline{H'G'}$, and $\overline{HG} \cong \overline{H''G''}$. $\overline{HB} \cong \overline{H'B}$. $\overline{GA} \cong \overline{G'A}$.
- Yes, $AC = BD$ because $\overline{GG''}$ and $\overline{HH''}$ are perpendicular to both k and m , so \overline{BD} and \overline{AC} are opposite sides of a rectangle.
- By the properties of reflections, $H'B = 9$ and $H'D = 4$. Theorem 9.5 implies that $GG'' = HH'' = 2 \cdot BD$, so the length of $\overline{GG''}$ is $2(9 + 4)$, or 26 units.



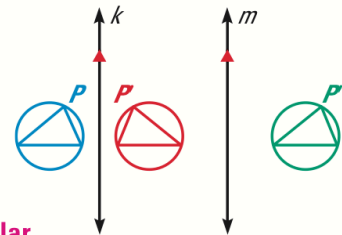
GUIDED PRACTICE for Examples 2 and 3

3. See margin for art; yes; the resulting segment $R'S'$ is not the same.

- Graph \overline{RS} from Example 2. Do the rotation first, followed by the reflection. Does the order of the transformations matter? *Explain.*
- In Example 3, part (c), *explain* how you know that $GG'' = HH''$.
They are opposite sides of a parallelogram.

Use the figure below for Exercises 5 and 6. The distance between line k and line m is 1.6 centimeters.

- The preimage is reflected in line k , then in line m . Describe a single transformation that maps the blue figure to the green figure. **translation**
- What is the distance between P and P'' ?
If you draw $\overline{PP''}$, what is its relationship with line k ? *Explain.* **3.2 cm; they are perpendicular.**



THEOREM

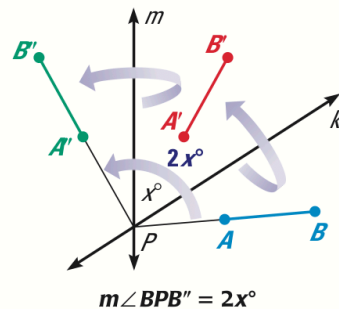
For Your Notebook

THEOREM 9.6 Reflections in Intersecting Lines Theorem

If lines k and m intersect at point P , then a reflection in k followed by a reflection in m is the same as a rotation about point P .

The angle of rotation is $2x^\circ$, where x° is the measure of the acute or right angle formed by k and m .

Proof: Ex. 38, p. 614



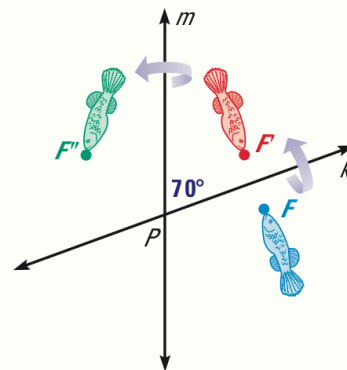
EXAMPLE 4 Use Theorem 9.6

In the diagram, the figure is reflected in line k . The image is then reflected in line m . Describe a single transformation that maps F to F'' .

Solution

The measure of the acute angle formed between lines k and m is 70° . So, by Theorem 9.6, a single transformation that maps F to F'' is a 140° rotation about point P .

You can check that this is correct by tracing lines k and m and point F , then rotating the point 140° .

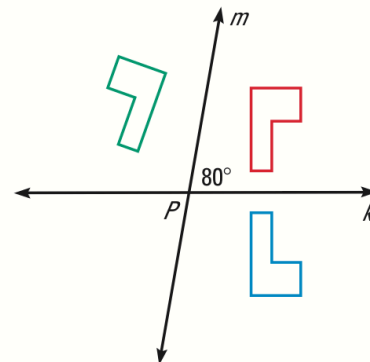


Animated Geometry at classzone.com

GUIDED PRACTICE for Example 4

7. a rotation of 160° about point P

- In the diagram at the right, the preimage is reflected in line k , then in line m . Describe a single transformation that maps the blue figure onto the green figure.
- A rotation of 76° maps C to C' . To map C to C' using two reflections, what is the angle formed by the intersecting lines of reflection? 38°



9.5 EXERCISES

HOMWORK KEY

○ = WORKED-OUT SOLUTIONS on p. WS1 for Exs. 7, 17, and 27

★ = STANDARDIZED TEST PRACTICE Exs. 2, 25, 29, and 34

SKILL PRACTICE

- A** 1. **VOCABULARY** Copy and complete: In a glide reflection, the direction of the translation must be ? to the line of reflection. **parallel**

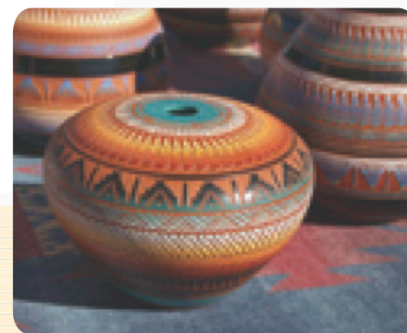
2. ★ **WRITING** Explain why a glide reflection is an isometry.
It preserves length and angle measure.

GLIDE REFLECTION The endpoints of \overline{CD} are $C(2, -5)$ and $D(4, 0)$. Graph the image of \overline{CD} after the glide reflection. 3–6. See margin.

- | | |
|---|---|
| 3. Translation: $(x, y) \rightarrow (x, y - 1)$
Reflection: in the y -axis | 4. Translation: $(x, y) \rightarrow (x - 3, y)$
Reflection: in $y = -1$ |
| 5. Translation: $(x, y) \rightarrow (x, y + 4)$
Reflection: in $x = 3$ | 6. Translation: $(x, y) \rightarrow (x + 2, y + 2)$
Reflection: in $y = x$ |

EXAMPLE 1
on p. 608
for Exs. 3–6

9.6 Identify Symmetry



Before

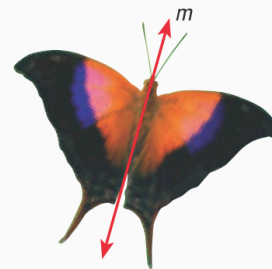
Now

Why?

Key Vocabulary

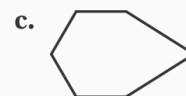
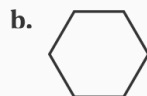
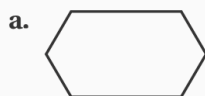
- line symmetry
- line of symmetry
- rotational symmetry
- center of symmetry

A figure in the plane has **line symmetry** if the figure can be mapped onto itself by a reflection in a line. This line of reflection is a **line of symmetry**, such as line m at the right. A figure can have more than one line of symmetry.



EXAMPLE 1 Identify lines of symmetry

How many lines of symmetry does the hexagon have?

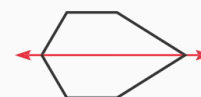
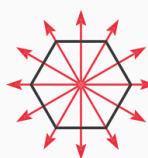
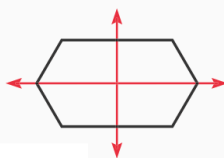


Solution

a. Two lines of symmetry

b. Six lines of symmetry

c. One line of symmetry



REVIEW REFLECTION

Notice that the lines of symmetry are also lines of reflection.

at classzone.com

GUIDED PRACTICE for Example 1

How many lines of symmetry does the object appear to have?



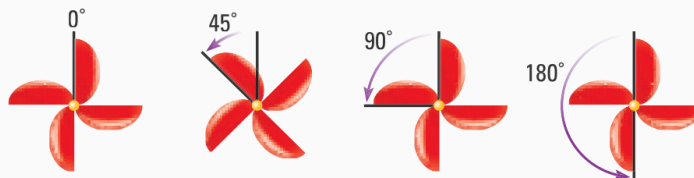
4. Draw a hexagon with no lines of symmetry. See margin.

ROTATIONAL SYMMETRY A figure in a plane has **rotational symmetry** if the figure can be mapped onto itself by a rotation of 180° or less about the center of the figure. This point is the **center of symmetry**. Note that the rotation can be either clockwise or counterclockwise.

REVIEW ROTATION

For a figure with rotational symmetry, the *angle of rotation* is the smallest angle that maps the figure onto itself.

For example, the figure below has rotational symmetry, because a rotation of either 90° or 180° maps the figure onto itself (although a rotation of 45° does not).



The figure above also has *point symmetry*, which is 180° rotational symmetry.

EXAMPLE 2 Identify rotational symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

a. Parallelogram



b. Regular octagon



c. Trapezoid



Solution

a. The parallelogram has rotational symmetry. The center is the intersection of the diagonals. A 180° rotation about the center maps the parallelogram onto itself.



b. The regular octagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of 45° , 90° , 135° , or 180° about the center all map the octagon onto itself.



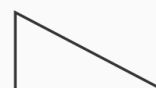
c. The trapezoid does not have rotational symmetry because no rotation of 180° or less maps the trapezoid onto itself.



GUIDED PRACTICE for Example 2

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

5. Rhombus **yes; 180° about the center** 6. Octagon **yes; 90° or 180° about the center** 7. Right triangle **no**





EXAMPLE 3 Standardized Test Practice

Identify the line symmetry and rotational symmetry of the equilateral triangle at the right.

- (A) 3 lines of symmetry, 60° rotational symmetry
- (B) 3 lines of symmetry, 120° rotational symmetry
- (C) 1 line of symmetry, 180° rotational symmetry
- (D) 1 line of symmetry, no rotational symmetry



ELIMINATE CHOICES

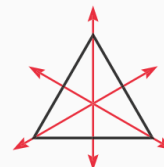
An equilateral triangle can be mapped onto itself by reflecting over any of three different lines. So, you can eliminate choices C and D.

Solution

The triangle has line symmetry. Three lines of symmetry can be drawn for the figure.

For a figure with s lines of symmetry, the smallest rotation that maps the figure onto itself has the measure $\frac{360^\circ}{s}$. So, the equilateral triangle has $\frac{360^\circ}{3}$, or 120° rotational symmetry.

▶ The correct answer is B. (A) (B) (C) (D)



GUIDED PRACTICE for Example 3

8. Describe the lines of symmetry and rotational symmetry of a non-equilateral isosceles triangle.
the altitude, no rotational symmetry

9.6 EXERCISES

HOMEWORK KEY

○ = WORKED-OUT SOLUTIONS

★ = STANDARDIZED TEST PRACTICE

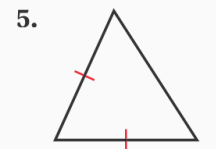
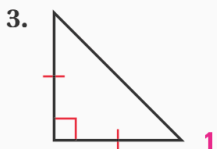
SKILL PRACTICE

- (A) 1. **VOCABULARY** What is a *center of symmetry*?
If a figure has rotational symmetry it is the point about which the figure is rotated.
2. ★ **WRITING** Draw a figure that has one line of symmetry and does not have rotational symmetry. Can a figure have two lines of symmetry and no rotational symmetry? See margin for art; no.

EXAMPLE 1

on p. 619
for Exs. 3–5

LINE SYMMETRY How many lines of symmetry does the triangle have?



9.7 Identify and Perform Dilations



Before

You used a coordinate rule to draw a dilation.

Now

You will use drawing tools and matrices to draw dilations.

Why?

So you can determine the scale factor of a photo, as in Ex. 37.

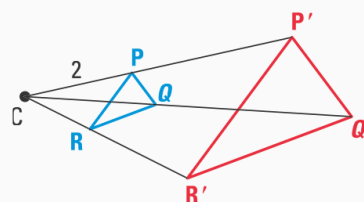
Key Vocabulary

- **scalar multiplication**
- **dilation**, p. 409
- **reduction**, p. 409
- **enlargement**, p. 409

Recall from Lesson 6.7 that a dilation is a transformation in which the original figure and its image are similar.

A dilation with center C and scale factor k maps every point P in a figure to a point P' so that one of the following statements is true:

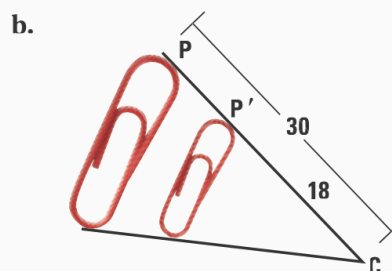
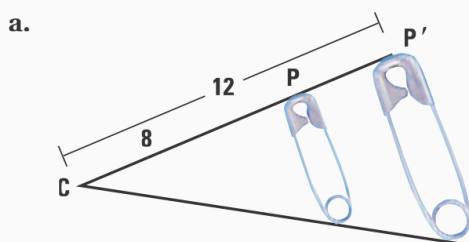
- If P is not the center point C , then the image point P' lies on \overrightarrow{CP} . The scale factor k is a positive number such that $k = \frac{CP'}{CP}$ and $k \neq 1$, or
- If P is the center point C , then $P = P'$.



As you learned in Lesson 6.7, the dilation is a *reduction* if $0 < k < 1$ and it is an *enlargement* if $k > 1$.

EXAMPLE 1 Identify dilations

Find the scale factor of the dilation. Then tell whether the dilation is a *reduction* or an *enlargement*.



Solution

a. Because $\frac{CP'}{CP} = \frac{12}{8}$, the scale factor is $k = \frac{3}{2}$. The image P' is an enlargement.

b. Because $\frac{CP'}{CP} = \frac{18}{30}$, the scale factor is $k = \frac{3}{5}$. The image P' is a reduction.

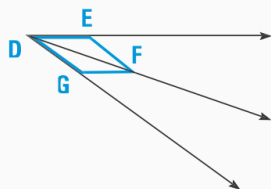
at classzone.com

EXAMPLE 2 Draw a dilation

Draw and label $\square DEFG$. Then construct a dilation of $\square DEFG$ with point D as the center of dilation and a scale factor of 2.

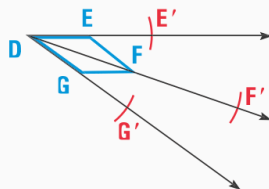
Solution

STEP 1



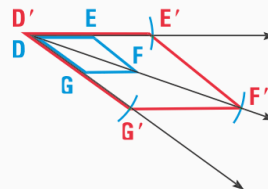
Draw $DEFG$. Draw rays from D through vertices E , F , and G .

STEP 2



Open the compass to the length of \overline{DE} . Locate E' on \overrightarrow{DE} so $DE' = 2(DE)$. Locate F' and G' the same way.

STEP 3



Add a second label D' to point D . Draw the sides of $D'E'F'G'$.



GUIDED PRACTICE for Examples 1 and 2

- In a dilation, $CP' = 3$ and $CP = 12$. Tell whether the dilation is a *reduction* or an *enlargement* and find its scale factor. **reduction, $\frac{1}{4}$**
- Draw and label $\triangle RST$. Then construct a dilation of $\triangle RST$ with R as the center of dilation and a scale factor of 3. **See margin.**

MATRICES **Scalar multiplication** is the process of multiplying each element of a matrix by a real number or *scalar*.

EXAMPLE 3 Scalar multiplication

Simplify the product: $4 \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix}$.

Solution

$$4 \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 4(3) & 4(0) & 4(1) \\ 4(2) & 4(-1) & 4(-3) \end{bmatrix} \quad \begin{array}{l} \text{Multiply each element} \\ \text{in the matrix by 4.} \end{array}$$

$$= \begin{bmatrix} 12 & 0 & 4 \\ 8 & -4 & -12 \end{bmatrix} \quad \text{Simplify.}$$



GUIDED PRACTICE for Example 3

Simplify the product.

$$3. \ 5 \begin{bmatrix} 2 & 1 & -10 \\ 3 & -4 & 7 \end{bmatrix} \begin{bmatrix} 10 & 5 & -50 \\ 15 & -20 & 35 \end{bmatrix} \quad 4. \ -2 \begin{bmatrix} -4 & 1 & 0 \\ 9 & -5 & -7 \end{bmatrix} \begin{bmatrix} 8 & -2 & 0 \\ -18 & 10 & 14 \end{bmatrix}$$

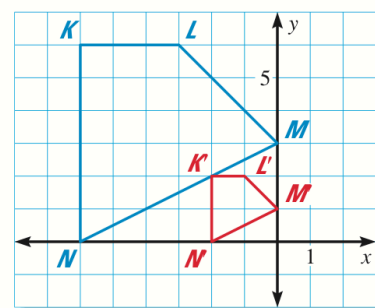
DILATIONS USING MATRICES You can use scalar multiplication to represent a dilation centered at the origin in the coordinate plane. To find the image matrix for a dilation centered at the origin, use the scale factor as the scalar.

EXAMPLE 4 Use scalar multiplication in a dilation

The vertices of quadrilateral $KLMN$ are $K(-6, 6)$, $L(-3, 6)$, $M(0, 3)$, and $N(-6, 0)$. Use scalar multiplication to find the image of $KLMN$ after a dilation with its center at the origin and a scale factor of $\frac{1}{3}$. Graph $KLMN$ and its image.

Solution

$$\begin{array}{l} \text{Scale} \\ \text{factor} \end{array} \frac{1}{3} \begin{array}{c} \text{Polygon} \\ \text{matrix} \end{array} \begin{array}{cccc} K & L & M & N \\ \begin{bmatrix} -6 & -3 & 0 & -6 \\ 6 & 6 & 3 & 0 \end{bmatrix} \end{array} = \begin{array}{cccc} \text{Image} \\ \text{matrix} \end{array} \begin{array}{cccc} K' & L' & M' & N' \\ \begin{bmatrix} -2 & -1 & 0 & -2 \\ 2 & 2 & 1 & 0 \end{bmatrix} \end{array}$$



EXAMPLE 5 Find the image of a composition

The vertices of $\triangle ABC$ are $A(-4, 1)$, $B(-2, 2)$, and $C(-2, 1)$. Find the image of $\triangle ABC$ after the given composition.

Translation: $(x, y) \rightarrow (x + 5, y + 1)$

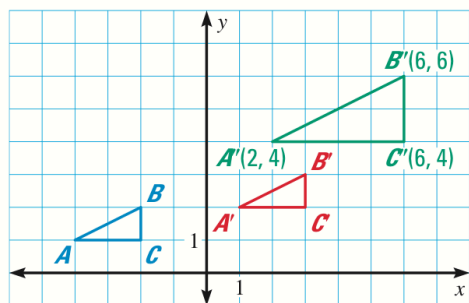
Dilation: centered at the origin with a scale factor of 2

Solution

STEP 1 Graph the preimage $\triangle ABC$ on the coordinate plane.

STEP 2 Translate $\triangle ABC$ 5 units to the right and 1 unit up. Label it $\triangle A'B'C'$.

STEP 3 Dilate $\triangle A'B'C'$ using the origin as the center and a scale factor of 2 to find $\triangle A''B''C''$.



✓ GUIDED PRACTICE for Examples 4 and 5

- The vertices of $\triangle RST$ are $R(1, 2)$, $S(2, 1)$, and $T(2, 2)$. Use scalar multiplication to find the vertices of $\triangle R'S'T'$ after a dilation with its center at the origin and a scale factor of 2. $R'(2, 4)$, $S'(4, 2)$, $T'(4, 4)$
- A segment has the endpoints $C(-1, 1)$ and $D(1, 1)$. Find the image of \overline{CD} after a 90° rotation about the origin followed by a dilation with its center at the origin and a scale factor of 2. $C'(-2, -2)$, $D'(-2, 2)$