8.1 Find Angle Measures in Polygons

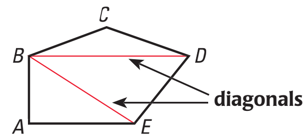
Before

Now

Why?

Key Vocabulary
- **diagonal**
- **interior angle,** *p. 218*
- **exterior angle,** *p. 218*

In a polygon, two vertices that are endpoints of the same side are called *consecutive vertices*. A **diagonal** of a polygon is a segment that joins two *nonconsecutive vertices*. Polygon *ABCDE* has two diagonals from vertex *B*, $\overline{BD}$ and $\overline{BE}$.



As you can see, the diagonals from one vertex form triangles. In the Activity on page 506, you used these triangles to find the sum of the interior angle measures of a polygon. Your results support the following theorem and corollary.

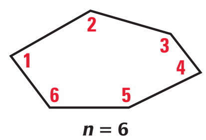
THEOREMS *For Your Notebook*

THEOREM 8.1 Polygon Interior Angles Theorem

The sum of the measures of the interior angles of a convex *n*-gon is $(n-2) \cdot 180°$.

$m\angle 1 + m\angle 2 + \cdots + m\angle n = (n-2) \cdot 180°$

Proof: Ex. 33, p. 512 (for pentagons)



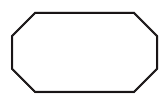
COROLLARY TO THEOREM 8.1 Interior Angles of a Quadrilateral

The sum of the measures of the interior angles of a quadrilateral is 360°.

Proof: Ex. 34, p. 512

EXAMPLE 1 Find the sum of angle measures in a polygon

Find the sum of the measures of the interior angles of a convex octagon.



Solution

An octagon has 8 sides. Use the Polygon Interior Angles Theorem.

$(n-2) \cdot 180° = (8-2) \cdot 180°$ **Substitute 8 for *n*.**

$= 6 \cdot 180°$ **Subtract.**

$= 1080°$ **Multiply.**

▶ The sum of the measures of the interior angles of an octagon is 1080°.

Resource Planning Guide

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EXAMPLE 2 Find the number of sides of a polygon

The sum of the measures of the interior angles of a convex polygon is 900°. Classify the polygon by the number of sides.

Solution

Use the Polygon Interior Angles Theorem to write an equation involving the number of sides n. Then solve the equation to find the number of sides.

$(n - 2) \cdot 180° = 900°$ — **Polygon Interior Angles Theorem**

$n - 2 = 5$ — **Divide each side by 180°.**

$n = 7$ — **Add 2 to each side.**

▶ The polygon has 7 sides. It is a heptagon.

✓ GUIDED PRACTICE for Examples 1 and 2

1. The coin shown is in the shape of a regular 11-gon. Find the sum of the measures of the interior angles. **1620°**

2. The sum of the measures of the interior angles of a convex polygon is 1440°. Classify the polygon by the number of sides. **decagon**

EXAMPLE 3 Find an unknown interior angle measure

ALGEBRA Find the value of x in the diagram shown.

Solution

The polygon is a quadrilateral. Use the Corollary to the Polygon Interior Angles Theorem to write an equation involving x. Then solve the equation.

$x° + 108° + 121° + 59° = 360°$ — **Corollary to Theorem 8.1**

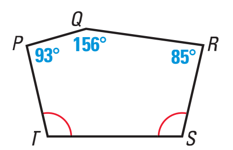
$x + 288 = 360$ — **Combine like terms.**

$x = 72$ — **Subtract 288 from each side.**

▶ The value of x is 72.

✓ GUIDED PRACTICE for Example 3

3. Use the diagram at the right. Find $m\angle S$ and $m\angle T$. **103°, 103°**

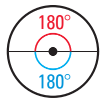


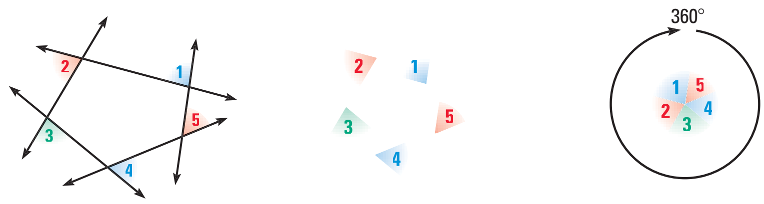
4. The measures of three of the interior angles of a quadrilateral are 89°, 110°, and 46°. Find the measure of the fourth interior angle. **115°**

Differentiated Instruction

EXTERIOR ANGLES Unlike the sum of the interior angle measures of a convex polygon, the sum of the exterior angle measures does *not* depend on the number of sides of the polygon. The diagrams below suggest that the sum of the measures of the exterior angles, one at each vertex, of a pentagon is 360°. In general, this sum is 360° for any convex polygon.

VISUALIZE IT
A circle contains two straight angles. So, there are 180° + 180°, or 360°, in a circle.





STEP 1 **Shade** one exterior angle at each vertex.

STEP 2 **Cut** out the exterior angles.

STEP 3 **Arrange** the exterior angles to form 360°.

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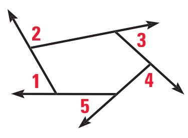
THEOREM *For Your Notebook*

THEOREM 8.2 Polygon Exterior Angles Theorem

The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360°.

$m\angle 1 + m\angle 2 + \cdots + m\angle n = 360°$

Proof: Ex. 35, p. 512



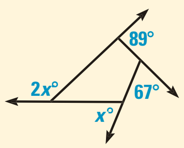
$n = 5$

EXAMPLE 4 Standardized Test Practice

What is the value of *x* in the diagram shown?

Ⓐ 67 Ⓑ 68

Ⓒ 91 Ⓓ 136



ELIMINATE CHOICES
You can quickly eliminate choice *D*. If *x* were equal to 136, then the sum of only two of the angle measures ($x°$ and $2x°$) would be greater than 360°.

Solution

Use the Polygon Exterior Angles Theorem to write and solve an equation.

$x° + 2x° + 89° + 67° = 360°$ **Polygon Exterior Angles Theorem**

$3x + 156 = 360$ **Combine like terms.**

$x = 68$ **Solve for *x*.**

▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

✓ GUIDED PRACTICE for Example 4

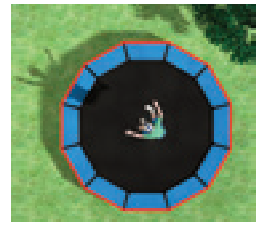
5. A convex hexagon has exterior angles with measures 34°, 49°, 58°, 67°, and 75°. What is the measure of an exterior angle at the sixth vertex? **77°**

Differentiated Instruction

EXAMPLE 5 Find angle measures in regular polygons

READ VOCABULARY
Recall that a *dodecagon* is a polygon with 12 sides and 12 vertices.

TRAMPOLINE The trampoline shown is shaped like a regular dodecagon. Find (a) the measure of each interior angle and (b) the measure of each exterior angle.



Solution

a. Use the Polygon Interior Angles Theorem to find the sum of the measures of the interior angles.

$$(n - 2) \cdot 180° = (12 - 2) \cdot 180° = 1800°$$

Then find the measure of one interior angle. A regular dodecagon has 12 congruent interior angles. Divide 1800° by 12: 1800° ÷ 12 = 150°.

▶ The measure of each interior angle in the dodecagon is 150°.

b. By the Polygon Exterior Angles Theorem, the sum of the measures of the exterior angles, one angle at each vertex, is 360°. Divide 360° by 12 to find the measure of one of the 12 congruent exterior angles: 360° ÷ 12 = 30°.

▶ The measure of each exterior angle in the dodecagon is 30°.

✓ GUIDED PRACTICE for Example 5

6. An interior angle and an adjacent exterior angle of a polygon form a linear pair. How can you use this fact as another method to find the exterior angle measure in Example 5? **Linear pairs are supplementary. Since the interior angle measures 150°, the exterior angle must measure 30°.**

8.1 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 9, 11, and 29

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 18, 23, and 37

◆ = **MULTIPLE REPRESENTATIONS** Ex. 36

8.2 Use Properties of Parallelograms

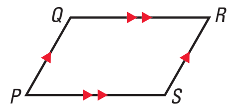
Before

Now

Why?

Key Vocabulary
- **parallelogram**

A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. The term "parallelogram *PQRS*" can be written as ▱*PQRS*. In ▱*PQRS*, $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{PS}$ by definition. The theorems below describe other properties of parallelograms.



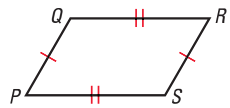
THEOREMS — *For Your Notebook*

THEOREM 8.3

If a quadrilateral is a parallelogram, then its opposite sides are congruent.

If *PQRS* is a parallelogram, then $\overline{PQ} \cong \overline{RS}$ and $\overline{QR} \cong \overline{PS}$.

Proof: p. 516

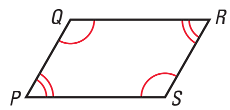


THEOREM 8.4

If a quadrilateral is a parallelogram, then its opposite angles are congruent.

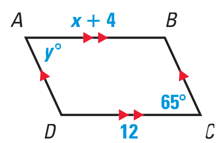
If *PQRS* is a parallelogram, then $\angle P \cong \angle R$ and $\angle Q \cong \angle S$.

Proof: Ex. 42, p. 520



EXAMPLE 1 Use properties of parallelograms

ALGEBRA Find the values of *x* and *y*.



ABCD is a parallelogram by the definition of a parallelogram. Use Theorem 8.3 to find the value of *x*.

$AB = CD$	Opposite sides of a ▱ are ≅.
$x + 4 = 12$	Substitute $x + 4$ for *AB* and 12 for *CD*.
$x = 8$	Subtract 4 from each side.

By Theorem 8.4, $\angle A \cong \angle C$, or $m\angle A = m\angle C$. So, $y° = 65°$.

▶ In ▱*ABCD*, $x = 8$ and $y = 65$.

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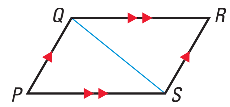
PROOF Theorem 8.3

If a quadrilateral is a parallelogram, then its opposite sides are congruent.



GIVEN ▶ *PQRS* is a parallelogram.

PROVE ▶ $\overline{PQ} \cong \overline{RS}$, $\overline{QR} \cong \overline{PS}$

Plan for Proof

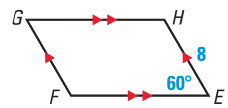
a. Draw diagonal $\overline{QS}$ to form $\triangle PQS$ and $\triangle RSQ$.
b. Use the ASA Congruence Postulate to show that $\triangle PQS \cong \triangle RSQ$.
c. Use congruent triangles to show that $\overline{PQ} \cong \overline{RS}$ and $\overline{QR} \cong \overline{PS}$.

Plan in Action

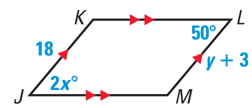
	STATEMENTS	REASONS
a.	**1.** *PQRS* is a ▱.	**1.** Given
	2. Draw $\overline{QS}$.	**2.** Through any 2 points there exists exactly 1 line.
	3. $\overline{PQ} \parallel \overline{RS}$, $\overline{QR} \parallel \overline{PS}$	**3.** Definition of parallelogram
b.	**4.** $\angle PQS \cong \angle RSQ$, $\angle PSQ \cong \angle RQS$	**4.** Alternate Interior Angles Theorem
	5. $\overline{QS} \cong \overline{QS}$	**5.** Reflexive Property of Congruence
	6. $\triangle PQS \cong \triangle RSQ$	**6.** ASA Congruence Postulate
c.	**7.** $\overline{PQ} \cong \overline{RS}$, $\overline{QR} \cong \overline{PS}$	**7.** Corresp. parts of ≅ ⚠ are ≅.

✓ GUIDED PRACTICE for Example 1

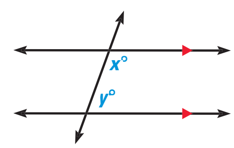
1. Find *FG* and $m\angle G$. **8, 60°**



2. Find the values of *x* and *y*. **25, 15**



INTERIOR ANGLES The Consecutive Interior Angles Theorem (page 155) states that if two parallel lines are cut by a transversal, then the pairs of consecutive interior angles formed are supplementary.



$x° + y° = 180°$

A pair of consecutive angles in a parallelogram are like a pair of consecutive interior angles between parallel lines. This similarity suggests Theorem 8.5.

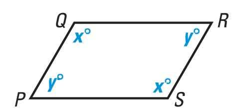
THEOREM *For Your Notebook*

THEOREM 8.5

If a quadrilateral is a parallelogram, then its consecutive angles are supplementary.

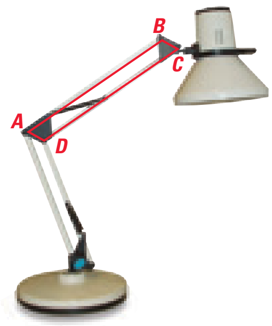
If *PQRS* is a parallelogram, then $x° + y° = 180°$.

Proof: Ex. 43, p. 520



EXAMPLE 2 Use properties of a parallelogram

DESK LAMP As shown, part of the extending arm of a desk lamp is a parallelogram. The angles of the parallelogram change as the lamp is raised and lowered. Find $m\angle BCD$ when $m\angle ADC = 110°$.



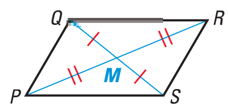
Solution

By Theorem 8.5, the consecutive angle pairs in $\square ABCD$ are supplementary. So, $m\angle ADC + m\angle BCD = 180°$. Because $m\angle ADC = 110°$, $m\angle BCD = 180° - 110° = 70°$.

THEOREM *For Your Notebook*

THEOREM 8.6

If a quadrilateral is a parallelogram, then its diagonals bisect each other.



$\overline{QM} \cong \overline{SM}$ and $\overline{PM} \cong \overline{RM}$

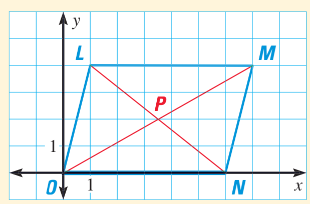
Proof: Ex. 44, p. 521

EXAMPLE 3 Standardized Test Practice

The diagonals of $\square LMNO$ intersect at point P. What are the coordinates of P?

Ⓐ $\left(\frac{7}{2}, 2\right)$　　Ⓑ $\left(2, \frac{7}{2}\right)$

Ⓒ $\left(\frac{5}{2}, 2\right)$　　Ⓓ $\left(2, \frac{5}{2}\right)$



Solution

By Theorem 8.6, the diagonals of a parallelogram bisect each other. So, P is the midpoint of diagonals $\overline{LN}$ and $\overline{OM}$. Use the Midpoint Formula.

$$\text{Coordinates of midpoint } P \text{ of } \overline{OM} = \left(\frac{7+0}{2}, \frac{4+0}{2}\right) = \left(\frac{7}{2}, 2\right)$$

▶ The correct answer is A. Ⓐ Ⓑ Ⓒ Ⓓ

SIMPLIFY CALCULATIONS

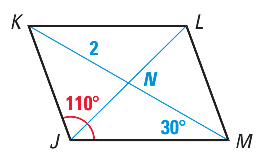
In Example 3, you can use either diagonal to find the coordinates of P. Using $\overline{OM}$ simplifies calculations because one endpoint is (0, 0).

✓ GUIDED PRACTICE for Examples 2 and 3

Find the indicated measure in $\square JKLM$.

3. NM 2　　4. KM 4

5. $m\angle JML$ 70°　　6. $m\angle KML$ 40°



8.3 Show that a Quadrilateral is a Parallelogram

Before

Now

Why?

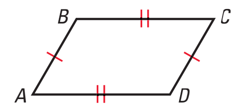
Key Vocabulary
- **parallelogram,** *p. 515*

Given a parallelogram, you can use Theorem 8.3 and Theorem 8.4 to prove statements about the angles and sides of the parallelogram. The converses of Theorem 8.3 and Theorem 8.4 are stated below. You can use these and other theorems in this lesson to prove that a quadrilateral with certain properties is a parallelogram.

THEOREMS — *For Your Notebook*

THEOREM 8.7

If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

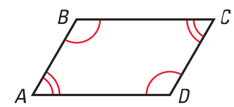


If $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$, then $ABCD$ is a parallelogram.

Proof: below

THEOREM 8.8

If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.



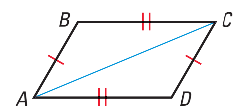
If $\angle A \cong \angle C$ and $\angle B \cong \angle D$, then $ABCD$ is a parallelogram.

Proof: Ex. 38, p. 529

PROOF Theorem 8.7

GIVEN ▶ $\overline{AB} \cong \overline{CD}$, $\overline{BC} \cong \overline{AD}$

PROVE ▶ $ABCD$ is a parallelogram.



Proof Draw $\overline{AC}$, forming $\triangle ABC$ and $\triangle CDA$. You are given that $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$. Also, $\overline{AC} \cong \overline{AC}$ by the Reflexive Property of Congruence. So, $\triangle ABC \cong \triangle CDA$ by the SSS Congruence Postulate. Because corresponding parts of congruent triangles are congruent, $\angle BAC \cong \angle DCA$ and $\angle BCA \cong DAC$. Then, by the Alternate Interior Angles Converse, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$. By definition, $ABCD$ is a parallelogram.

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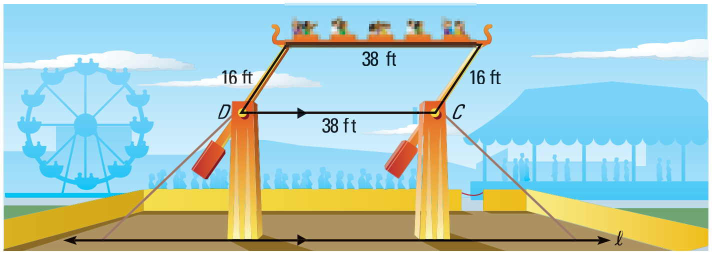
EXAMPLE 1 Solve a real-world problem

RIDE An amusement park ride has a moving platform attached to four swinging arms. The platform swings back and forth, higher and higher, until it goes over the top and around in a circular motion. In the diagram below, $\overline{AD}$ and $\overline{BC}$ represent two of the swinging arms, and $\overline{DC}$ is parallel to the ground (line ℓ). *Explain* why the moving platform $\overline{AB}$ is always parallel to the ground.



Solution

The shape of quadrilateral *ABCD* changes as the moving platform swings around, but its side lengths do not change. Both pairs of opposite sides are congruent, so *ABCD* is a parallelogram by Theorem 8.7.

By the definition of a parallelogram, $\overline{AB} \parallel \overline{DC}$. Because $\overline{DC}$ is parallel to line ℓ, $\overline{AB}$ is also parallel to line ℓ by the Transitive Property of Parallel Lines. So, the moving platform is parallel to the ground.

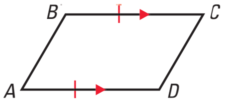
✓ GUIDED PRACTICE for Example 1

1. In quadrilateral *WXYZ*, $m\angle W = 42°$, $m\angle X = 138°$, $m\angle Y = 42°$. Find $m\angle Z$. Is *WXYZ* a parallelogram? *Explain* your reasoning. **138°; yes; the sum of the measures of the interior angles in a quadrilateral is 360°, so the measure of $\angle Z$ is 138°. Since opposite angles of the quadrilateral are congruent, *WXYZ* is a parallelogram.**

THEOREMS *For Your Notebook*

THEOREM 8.9

If one pair of opposite sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.

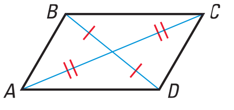


If $\overline{BC} \parallel \overline{AD}$ and $\overline{BC} \cong \overline{AD}$, then *ABCD* is a parallelogram.

Proof: Ex. 33, p. 528

THEOREM 8.10

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



If $\overline{BD}$ and $\overline{AC}$ bisect each other, then *ABCD* is a parallelogram.

Proof: Ex. 39, p. 529

EXAMPLE 2 Identify a parallelogram

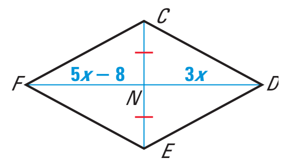
ARCHITECTURE The doorway shown is part of a building in England. Over time, the building has leaned sideways. *Explain* how you know that $SV = TU$.

Solution

In the photograph, $\overline{ST} \parallel \overline{UV}$ and $\overline{ST} \cong \overline{UV}$. By Theorem 8.9, quadrilateral $STUV$ is a parallelogram. By Theorem 8.3, you know that opposite sides of a parallelogram are congruent. So, $SV = TU$.

EXAMPLE 3 Use algebra with parallelograms

xy ALGEBRA For what value of x is quadrilateral $CDEF$ a parallelogram?



Solution

By Theorem 8.10, if the diagonals of $CDEF$ bisect each other, then it is a parallelogram. You are given that $\overline{CN} \cong \overline{EN}$. Find x so that $\overline{FN} \cong \overline{DN}$.

$FN = DN$	**Set the segment lengths equal.**
$5x - 8 = 3x$	**Substitute $5x - 8$ for FN and $3x$ for DN.**
$2x - 8 = 0$	**Subtract $3x$ from each side.**
$2x = 8$	**Add 8 to each side.**
$x = 4$	**Divide each side by 2.**

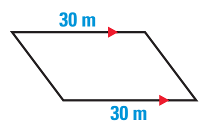
When $x = 4$, $FN = 5(4) - 8 = 12$ and $DN = 3(4) = 12$.

▶ Quadrilateral $CDEF$ is a parallelogram when $x = 4$.

✓ GUIDED PRACTICE for Examples 2 and 3

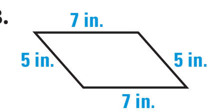
What theorem can you use to show that the quadrilateral is a parallelogram?

2.



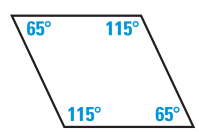
Theorem 8.9

3.



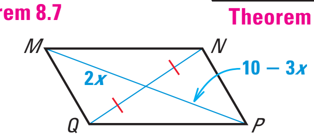
Theorem 8.7

4.



Theorem 8.8

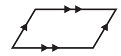
5. For what value of x is quadrilateral $MNPQ$ a parallelogram? *Explain* your reasoning. **2; the diagonals of a parallelogram bisect each other so solve $2x = 10 - 3x$ for x.**



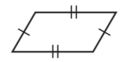
CONCEPT SUMMARY *For Your Notebook*

Ways to Prove a Quadrilateral is a Parallelogram

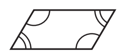
1. Show both pairs of opposite sides are parallel.
(DEFINITION)



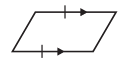
2. Show both pairs of opposite sides are congruent.
(THEOREM 8.7)



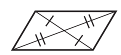
3. Show both pairs of opposite angles are congruent.
(THEOREM 8.8)



4. Show one pair of opposite sides are congruent and parallel.
(THEOREM 8.9)

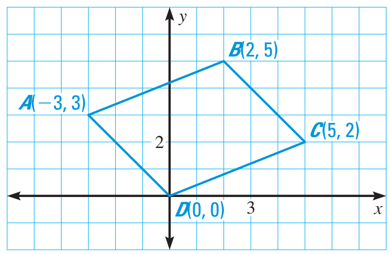


5. Show the diagonals bisect each other.
(THEOREM 8.10)



EXAMPLE 4 Use coordinate geometry

Show that quadrilateral *ABCD* is a parallelogram.



Solution

One way is to show that a pair of sides are congruent and parallel. Then apply Theorem 8.9.

First use the Distance Formula to show that $\overline{AB}$ and $\overline{CD}$ are congruent.

$$AB = \sqrt{[2 - (-3)]^2 + (5 - 3)^2} = \sqrt{29} \qquad CD = \sqrt{(5 - 0)^2 + (2 - 0)^2} = \sqrt{29}$$

Because $AB = CD = \sqrt{29}$, $\overline{AB} \cong \overline{CD}$.

Then use the slope formula to show that $\overline{AB} \parallel \overline{CD}$.

$$\text{Slope of } \overline{AB} = \frac{5 - (3)}{2 - (-3)} = \frac{2}{5} \qquad \text{Slope of } \overline{CD} = \frac{2 - 0}{5 - 0} = \frac{2}{5}$$

Because $\overline{AB}$ and $\overline{CD}$ have the same slope, they are parallel.

▶ $\overline{AB}$ and $\overline{CD}$ are congruent and parallel. So, *ABCD* is a parallelogram by Theorem 8.9.

ANOTHER WAY
For alternative methods for solving the problem in Example 4, turn to page 530 for the **Problem Solving Workshop**.

6. Find the slopes of all four sides and show that opposite sides are parallel. A second way is to find the length of each side and show that opposite sides are congruent. A third way is to find the point of intersection of the diagonals and show the diagonals bisect each other.

✓ GUIDED PRACTICE for Example 4

6. Refer to the Concept Summary above. *Explain* how other methods can be used to show that quadrilateral *ABCD* in Example 4 is a parallelogram.

8.4 Properties of Rhombuses, Rectangles, and Squares

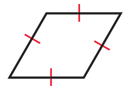
Before

Now

Why?

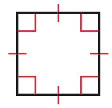
Key Vocabulary
- rhombus
- rectangle
- square

In this lesson, you will learn about three special types of parallelograms: *rhombuses*, *rectangles*, and *squares*.



A rhombus is a parallelogram with four congruent sides.

A rectangle is a parallelogram with four right angles.



A square is a parallelogram with four congruent sides and four right angles.

You can use the corollaries below to prove that a quadrilateral is a rhombus, rectangle, or square, without first proving that the quadrilateral is a parallelogram.

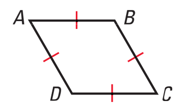
COROLLARIES — *For Your Notebook*

Rhombus Corollary

A quadrilateral is a rhombus if and only if it has four congruent sides.

ABCD is a rhombus if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

Proof: Ex. 57, p. 539

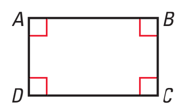


Rectangle Corollary

A quadrilateral is a rectangle if and only if it has four right angles.

ABCD is a rectangle if and only if $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof: Ex. 58, p. 539

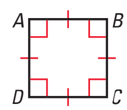


Square Corollary

A quadrilateral is a square if and only if it is a rhombus and a rectangle.

ABCD is a square if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$ and $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof: Ex. 59, p. 539



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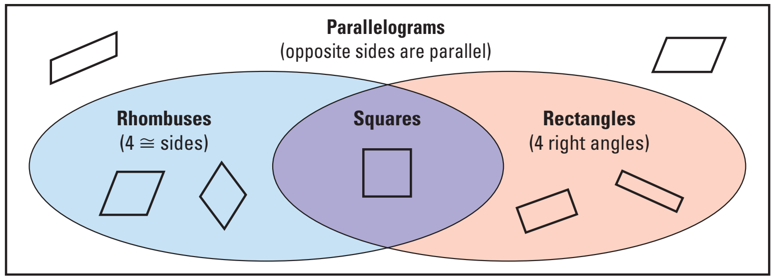
The *Venn diagram* below illustrates some important relationships among parallelograms, rhombuses, rectangles, and squares. For example, you can see that a square is a rhombus because it is a parallelogram with four congruent sides. Because it has four right angles, a square is also a rectangle.



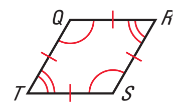
EXAMPLE 1 Use properties of special quadrilaterals

For any rhombus *QRST*, decide whether the statement is *always* or *sometimes* true. Draw a sketch and explain your reasoning.

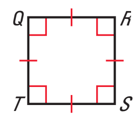
a. $\angle Q \cong \angle S$ **b.** $\angle Q \cong \angle R$

Solution

a. By definition, a rhombus is a parallelogram with four congruent sides. By Theorem 8.4, opposite angles of a parallelogram are congruent. So, $\angle Q \cong \angle S$. The statement is *always* true.

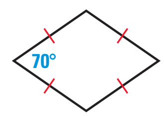


b. If rhombus *QRST* is a square, then all four angles are congruent right angles. So, $\angle Q \cong \angle R$ if *QRST* is a square. Because not all rhombuses are also squares, the statement is *sometimes* true.



EXAMPLE 2 Classify special quadrilaterals

Classify the special quadrilateral. Explain your reasoning.



Solution

The quadrilateral has four congruent sides. One of the angles is not a right angle, so the rhombus is not also a square. By the Rhombus Corollary, the quadrilateral is a rhombus.

✓ GUIDED PRACTICE for Examples 1 and 2

1. For any rectangle *EFGH*, is it *always* or *sometimes* true that $\overline{FG} \cong \overline{GH}$? *Explain* your reasoning. **Sometimes; this is only true if *EFGH* is a square.**
2. A quadrilateral has four congruent sides and four congruent angles. Sketch the quadrilateral and classify it. **See margin for art; square.**

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DIAGONALS The theorems below describe some properties of the diagonals of rhombuses and rectangles.

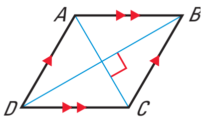
THEOREMS — *For Your Notebook*

THEOREM 8.11

A parallelogram is a rhombus if and only if its diagonals are perpendicular.

▱*ABCD* is a rhombus if and only if $\overline{AC} \perp \overline{BD}$.

Proof: p. 536; Ex. 56, p. 539

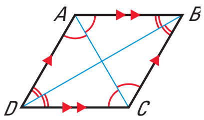


THEOREM 8.12

A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.

▱*ABCD* is a rhombus if and only if $\overline{AC}$ bisects $\angle BCD$ and $\angle BAD$ and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

Proof: Exs. 60–61, p. 539

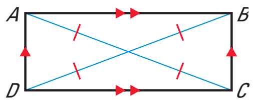


THEOREM 8.13

A parallelogram is a rectangle if and only if its diagonals are congruent.

▱*ABCD* is a rectangle if and only if $\overline{AC} \cong \overline{BD}$.

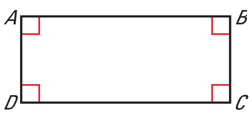
Proof: Exs. 63–64, p. 540



EXAMPLE 3 List properties of special parallelograms

Sketch rectangle *ABCD*. List everything that you know about it.

Solution



By definition, you need to draw a figure with the following properties:

- The figure is a parallelogram.
- The figure has four right angles.

Because *ABCD* is a parallelogram, it also has these properties:

- Opposite sides are parallel and congruent.
- Opposite angles are congruent. Consecutive angles are supplementary.
- Diagonals bisect each other.

By Theorem 8.13, the diagonals of *ABCD* are congruent.

Animated Geometry at classzone.com

3. See margin for art; *PQRS* is a parallelogram, rectangle, and a rhombus. Opposite pairs of sides are parallel and all four sides are congruent. All four angles are right angles. Diagonals are congruent and bisect each other.

✓ GUIDED PRACTICE for Example 3

3. Sketch square *PQRS*. List everything you know about the square.

Differentiated Instruction

BICONDITIONALS Recall that biconditionals such as Theorem 8.11 can be rewritten as two parts. To prove a biconditional, you must prove both parts.

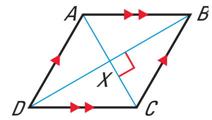
Conditional statement If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

Converse If a parallelogram is a rhombus, then its diagonals are perpendicular.

PROOF Part of Theorem 8.11

PROVE THEOREMS
You will prove the other part of Theorem 8.11 in Exercise 56 on page 539.

If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.



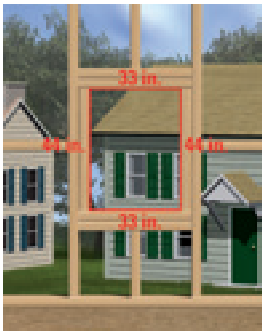
GIVEN ▶ *ABCD* is a parallelogram; $\overline{AC} \perp \overline{BD}$

PROVE ▶ *ABCD* is a rhombus.

Proof *ABCD* is a parallelogram, so $\overline{AC}$ and $\overline{BD}$ bisect each other, and $\overline{BX} \cong \overline{DX}$. Also, $\angle BXC$ and $\angle CXD$ are congruent right angles, and $\overline{CX} \cong \overline{CX}$. So, $\triangle BXC \cong \triangle DXC$ by the SAS Congruence Postulate. Corresponding parts of congruent triangles are congruent, so $\overline{BC} \cong \overline{DC}$. Opposite sides of a $\square ABCD$ are congruent, so $\overline{AD} \cong \overline{BC} \cong \overline{DC} \cong \overline{AB}$. By definition, *ABCD* is a rhombus.

EXAMPLE 4 Solve a real-world problem

CARPENTRY You are building a frame for a window. The window will be installed in the opening shown in the diagram.



a. The opening must be a rectangle. Given the measurements in the diagram, can you assume that it is? *Explain*.

b. You measure the diagonals of the opening. The diagonals are 54.8 inches and 55.3 inches. What can you conclude about the shape of the opening?

Solution

a. No, you cannot. The boards on opposite sides are the same length, so they form a parallelogram. But you do not know whether the angles are right angles.

b. By Theorem 8.13, the diagonals of a rectangle are congruent. The diagonals of the quadrilateral formed by the boards are not congruent, so the boards do not form a rectangle.

✓ GUIDED PRACTICE for Example 4

4. Suppose you measure only the diagonals of a window opening. If the diagonals have the same measure, can you conclude that the opening is a rectangle? *Explain*. **yes; Theorem 8.13**

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8.5 Use Properties of Trapezoids and Kites

Before

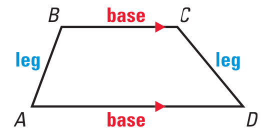
Now

Why?

Key Vocabulary
- **trapezoid**
  bases, base angles, legs
- **isosceles trapezoid**
- **midsegment of a trapezoid**
- **kite**

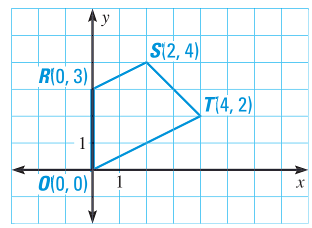
A **trapezoid** is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the **bases**.

A trapezoid has two pairs of **base angles**. For example, in trapezoid *ABCD*, $\angle A$ and $\angle D$ are one pair of base angles, and $\angle B$ and $\angle C$ are the second pair. The nonparallel sides are the **legs** of the trapezoid.



EXAMPLE 1 Use a coordinate plane

Show that *ORST* is a trapezoid.



Solution

Compare the slopes of opposite sides.

Slope of $\overline{RS} = \frac{4-3}{2-0} = \frac{1}{2}$

Slope of $\overline{OT} = \frac{2-0}{4-0} = \frac{2}{4} = \frac{1}{2}$

The slopes of $\overline{RS}$ and $\overline{OT}$ are the same, so $\overline{RS} \parallel \overline{OT}$.

Slope of $\overline{ST} = \frac{2-4}{4-2} = \frac{-2}{2} = -1$

Slope of $\overline{OR} = \frac{3-0}{0-0} = \frac{3}{0}$, which is undefined.

The slopes of $\overline{ST}$ and $\overline{OR}$ are not the same, so $\overline{ST}$ is not parallel to $\overline{OR}$.

▶ Because quadrilateral *ORST* has exactly one pair of parallel sides, it is a trapezoid.

✓ GUIDED PRACTICE for Example 1

1. Parallelogram; opposite pairs of sides are parallel.

1. **WHAT IF?** In Example 1, suppose the coordinates of point *S* are (4, 5). What type of quadrilateral is *ORST*? *Explain.*
2. In Example 1, which of the interior angles of quadrilateral *ORST* are supplementary angles? *Explain* your reasoning.
   $\angle O$ and $\angle R$, $\angle T$ and $\angle S$; Consecutive Interior Angles Theorem

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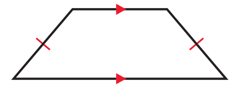
ISOSCELES TRAPEZOIDS If the legs of a trapezoid are congruent, then the trapezoid is an **isosceles trapezoid**.



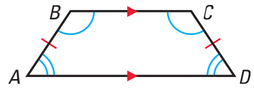
isosceles trapezoid

THEOREMS *For Your Notebook*

THEOREM 8.14

If a trapezoid is isosceles, then each pair of base angles is congruent.

If trapezoid *ABCD* is isosceles, then $\angle A \cong \angle D$ and $\angle B \cong \angle C$.

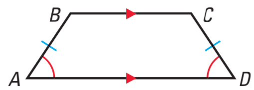


Proof: Ex. 37, p. 548

THEOREM 8.15

If a trapezoid has a pair of congruent base angles, then it is an isosceles trapezoid.

If $\angle A \cong \angle D$ (or if $\angle B \cong \angle C$), then trapezoid *ABCD* is isosceles.

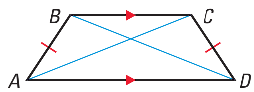


Proof: Ex. 38, p. 548

THEOREM 8.16

A trapezoid is isosceles if and only if its diagonals are congruent.

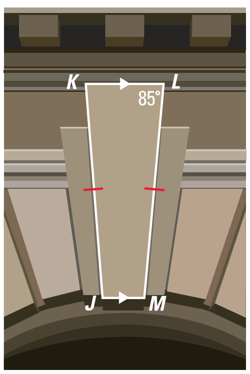
Trapezoid *ABCD* is isosceles if and only if $\overline{AC} \cong \overline{BD}$.



Proof: Exs. 39 and 43, p. 549

EXAMPLE 2 Use properties of isosceles trapezoids

ARCH The stone above the arch in the diagram is an isosceles trapezoid. Find $m\angle K$, $m\angle M$, and $m\angle J$.



Solution

STEP 1 **Find** $m\angle K$. *JKLM* is an isosceles trapezoid, so $\angle K$ and $\angle L$ are congruent base angles, and $m\angle K = m\angle L = 85°$.

STEP 2 **Find** $m\angle M$. Because $\angle L$ and $\angle M$ are consecutive interior angles formed by $\overleftrightarrow{LM}$ intersecting two parallel lines, they are supplementary. So, $m\angle M = 180° - 85° = 95°$.

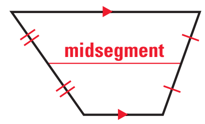
STEP 3 **Find** $m\angle J$. Because $\angle J$ and $\angle M$ are a pair of base angles, they are congruent, and $m\angle J = m\angle M = 95°$.

▶ So, $m\angle J = 95°$, $m\angle K = 85°$, and $m\angle M = 95°$.

Differentiated Instruction

READ VOCABULARY
The midsegment of a trapezoid is sometimes called the *median* of the trapezoid.

MIDSEGMENTS Recall that a midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. The _______ _______ is the segment that connects the midpoints of its legs.

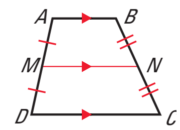


The theorem below is similar to the Midsegment Theorem for Triangles.

THEOREM *For Your Notebook*

THEOREM 8.17 Midsegment Theorem for Trapezoids

The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

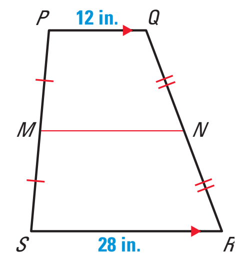


If $\overline{MN}$ is the midsegment of trapezoid *ABCD*, then $\overline{MN} \parallel \overline{AB}$, $\overline{MN} \parallel \overline{DC}$, and $MN = \frac{1}{2}(AB + CD)$.

Justification: Ex. 40, p. 549
Proof: p. 937

EXAMPLE 3 Use the midsegment of a trapezoid

In the diagram, $\overline{MN}$ is the midsegment of trapezoid *PQRS*. Find *MN*.



Solution

Use Theorem 8.17 to find *MN*.

$MN = \frac{1}{2}(PQ + SR)$ **Apply Theorem 8.17.**

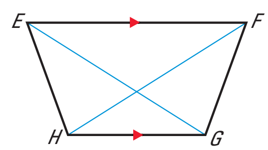
$= \frac{1}{2}(12 + 28)$ **Substitute 12 for *PQ* and 28 for *XU*.**

$= 20$ **Simplify.**

▶ The length *MN* is 20 inches.

✓ GUIDED PRACTICE for Examples 2 and 3

In Exercises 3 and 4, use the diagram of trapezoid *EFGH*.



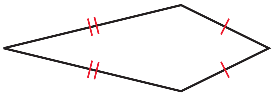
3. If $EG = FH$, is trapezoid *EFGH* isosceles? *Explain.* **yes; Theorem 8.16**

4. If $m\angle HEF = 70°$ and $m\angle FGH = 110°$, is trapezoid *EFGH* isosceles? *Explain.*

4. Yes. *Sample answer:* $m\angle EFG = 70°$ by Consecutive Interior Angles Theorem making *EFGH* an isosceles trapezoid by Theorem 8.15.

5. In trapezoid *JKLM*, $\angle J$ and $\angle M$ are right angles, and $JK = 9$ cm. The length of the midsegment $\overline{NP}$ of trapezoid *JKLM* is 12 cm. Sketch trapezoid *JKLM* and its midsegment. Find *ML*. *Explain* your reasoning.
See margin for art; 15 cm; solve $\frac{1}{2}(9 + x) = 12$ for x to find *ML*.

KITES A is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent.



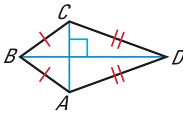
THEOREMS — *For Your Notebook*

THEOREM 8.18

If a quadrilateral is a kite, then its diagonals are perpendicular.

If quadrilateral $ABCD$ is a kite, then $\overline{AC} \perp \overline{BD}$.

Proof: Ex. 41, p. 549

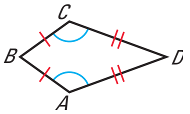


THEOREM 8.19

If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

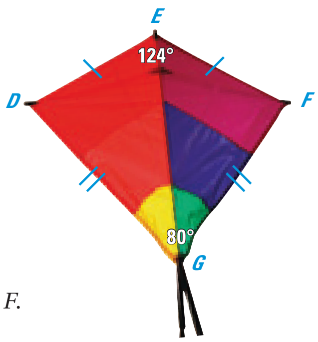
If quadrilateral $ABCD$ is a kite and $\overline{BC} \cong \overline{BA}$, then $\angle A \cong \angle C$ and $\angle B \not\cong \angle D$.

Proof: Ex. 42, p. 549



EXAMPLE 4 Apply Theorem 8.19

Find $m\angle D$ in the kite shown at the right.



Solution

By Theorem 8.19, $DEFG$ has exactly one pair of congruent opposite angles. Because $\angle E \not\cong \angle G$, $\angle D$ and $\angle F$ must be congruent. So, $m\angle D = m\angle F$. Write and solve an equation to find $m\angle D$.

$m\angle D + m\angle F + 124° + 80° = 360°$ **Corollary to Theorem 8.1**

$m\angle D + m\angle D + 124° + 80° = 360°$ **Substitute $m\angle D$ for $m\angle F$.**

$2(m\angle D) + 204° = 360°$ **Combine like terms.**

$m\angle D = 78°$ **Solve for $m\angle D$.**

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✓ GUIDED PRACTICE for Example 4

6. In a kite, the measures of the angles are $3x°$, 75°, 90°, and 120°. Find the value of x. What are the measures of the angles that are congruent? **25; 75°**

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8.6 Identify Special Quadrilaterals

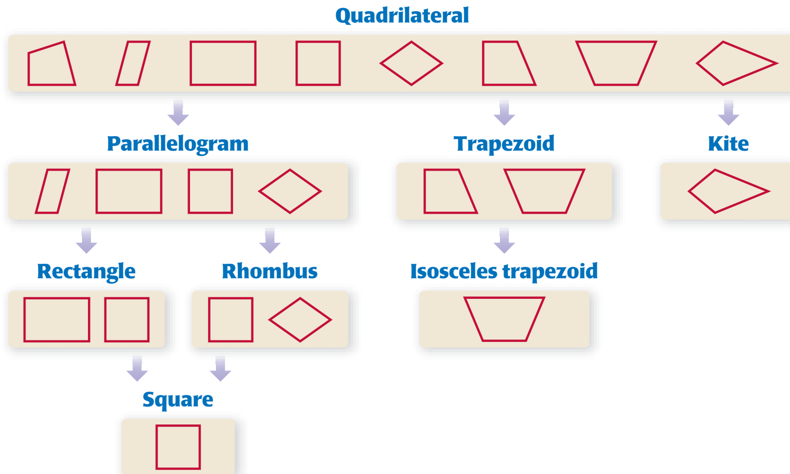
Before

Now

Why?

Key Vocabulary
- **parallelogram,** *p. 515*
- **rhombus,** *p. 533*
- **rectangle,** *p. 533*
- **square,** *p. 533*
- **trapezoid,** *p. 542*
- **kite,** *p. 545*

The diagram below shows relationships among the special quadrilaterals you have studied in Chapter 8. Each shape in the diagram has the properties of the shapes linked above it. For example, a rhombus has the properties of a parallelogram and a quadrilateral.

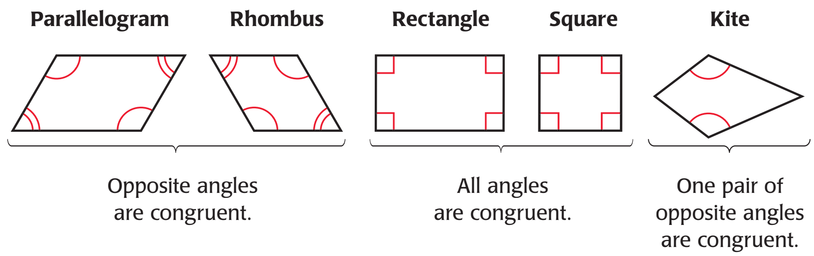


EXAMPLE 1 Identify quadrilaterals

Quadrilateral *ABCD* has at least one pair of opposite angles congruent. What types of quadrilaterals meet this condition?

Solution

There are many possibilities.



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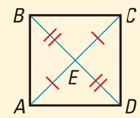
EXAMPLE 2 Standardized Test Practice

AVOID ERRORS
In Example 2, *ABCD* is shaped like a square. But you must rely only on marked information when you interpret a diagram.

What is the most specific name for quadrilateral *ABCD*?

Ⓐ Parallelogram Ⓑ Rhombus

Ⓒ Square Ⓓ Rectangle



Solution

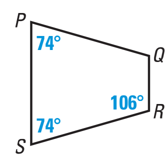
The diagram shows $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$. So, the diagonals bisect each other. By Theorem 8.10, *ABCD* is a parallelogram.

Rectangles, rhombuses and squares are also parallelograms. However, there is no information given about the side lengths or angle measures of *ABCD*. So, you cannot determine whether it is a rectangle, a rhombus, or a square.

▸ The correct answer is A. Ⓐ Ⓑ Ⓒ Ⓓ

EXAMPLE 3 Identify a quadrilateral

Is enough information given in the diagram to show that quadrilateral *PQRS* is an isosceles trapezoid? Explain.



Solution

STEP 1 **Show** that *PQRS* is a trapezoid. $\angle R$ and $\angle S$ are supplementary, but $\angle P$ and $\angle S$ are not. So, $\overline{PS} \parallel \overline{QR}$, but $\overline{PQ}$ is not parallel to $\overline{SR}$. By definition, *PQRS* is a trapezoid.

STEP 2 **Show** that trapezoid *PQRS* is isosceles. $\angle P$ and $\angle S$ are a pair of congruent base angles. So, *PQRS* is an isosceles trapezoid by Theorem 8.15.

▸ Yes, the diagram is sufficient to show that *PQRS* is an isosceles trapezoid.

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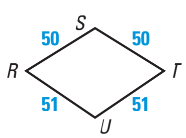
✓ GUIDED PRACTICE for Examples 1, 2, and 3

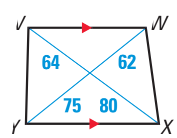
1. Quadrilateral *DEFG* has at least one pair of opposite sides congruent. What types of quadrilaterals meet this condition?
parallelogram, rectangle, square, rhombus, trapezoid

Give the most specific name for the quadrilateral. *Explain* your reasoning.

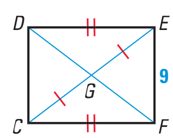
2.



3.



4.



2. Kite; there are two pair of consecutive congruent sides.

3. Trapezoid; there is one pair of parallel sides.

4. Quadrilateral; there is not enough information to be more specific.

5. ERROR ANALYSIS A student knows the following information about quadrilateral *MNPQ*: $\overline{MN} \parallel \overline{PQ}$, $\overline{MP} \cong \overline{NQ}$, and $\angle P \cong \angle Q$. The student concludes that *MNPQ* is an isosceles trapezoid. *Explain* why the student cannot make this conclusion. **It's possible that *MNPQ* could be a rectangle or a square since you don't know the relationship between $\overline{MQ}$ and $\overline{NP}$.**

Differentiated Instruction